Comparisons of Inclination-only Statistical Methods

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1 Abstract

Paleomagnetic data from borecores often lack declinations, and the arithmetic means of inclination-only data are known to introduce a shallowing bias. Several methods have been proposed to estimate unbiased means of the inclination along with measures of the precision.

Using maximum likelihood estimates, we were able to derive a robust technique of inclination-only statistics for the mean inclination and precision parameter, without making the assumptions and approximations of previous methods. Our method is described by Arason and Levi at the 2006 AGU Fall meeting.

To assess the reliability and accuracy of our method, we generated random Fisher-distributed data sets and used seven methods to estimate mean inclinations and precision parameters. We used true inclination values of 0, 10, ..., 80, and 90 degrees; true precision parameters of 10, 20, 40, and 100; the sample number in each data set was 5, 10, 20, and 100.

For each combination, we generated one thousand random Fisher-distributed data sets, and for these 160 000 data sets we calculated the true Fisher mean, also using declinations. For inclination-only data the mean was calculated using the following methods: Arithmetic mean; Kono (1980); McFadden-Reid (1982), using both their original and modified methods; Enkin-Watson (1996) gaussian-estimates; finally, we obtained maximum likelihood estimates by our new robust technique.

In many cases the estimates provided by the previous methods are significantly displaced from the true peak of the likelihood function to systematically shallower inclinations, especially for steep and dispersed data. It appears that the mean inclination estimates of the original McFadden-Reid statistics, still used by some paleomagnetists, is nearly identical to the arithmetic mean, and, in our opinion this method should be abandoned.

Comparisons of the results of the various methods is very favourable to our new maximum likelihood method. On average, it gives the most reliable estimates and the mean inclination estimates are the least biased toward shallow values. Further information on our inclination-only analysis, method, and program codes can be obtained from:

http://www.vedur.is/~arason/paleomag

3 The Importance of Unbiased Mean Inclinations

Forty years ago ***Briden and Ward*** (Pure Appl. Geophys., 63, 133-152, 1966) showed that for Fisher-distributed inclination-only data, the arithmetic mean is biased toward shallow inclinations.

In paleomagnetic applications this inclination bias is usually less than a few degrees. For individual studies such a discrepancy is of a minor importance and usually well within the confidence limits of the study. However, since this is a one sided bias, any attempt to combine results of many studies, in order to increase resolution, may lead to errors. For example, studies of long term non-dipole terms in the geomagnetic field require averaging the mean inclinations of many studies, indicating a long term effect of 1-2°. Improper procedures for estimating mean inclinations in individual studies can seriously affect such estimates.

In box A we show average inclination shallowing bias versus true inclination for various values of (κ). The Arithmetic mean and the McFadden-Reid methods give the worst biases. The Arason-Levi method results in the least biased estimates, and for $\kappa > 40$ it results in insignificant mean bias for inclinations up to 80°.

In box B we show the results of various methods for a particular numerical example. The Briden-Ward, Kono and McFadden-Reid methods all attempt to evaluate the maximum likelihood estimates. For this data set, only the Arason-Levi maximum likelihood method accurately calculates the location of the maximum value of the likelihood function.

In box C we show our comparison for one particular combination of true inclination, precision parameter and sample number (I = 70°, κ = 20, N = 100). For all our combinations we generated one thousand random Fisher-distributed data sets. In scatter plots we show the distribution of the estimates of the various methods (I, κ), then we show histograms of I and κ.

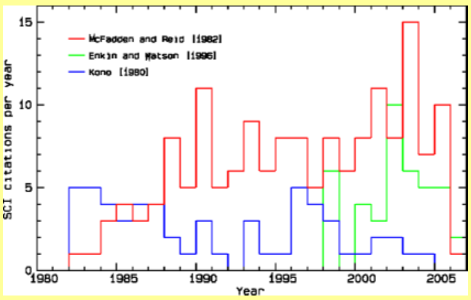
The information to separate the mean inclination from the precision parameter will be lost for very steep and dispersed data sets. For such data sets the likelihood function has a tendency to take its maximum value on the edge, i.e. at $I = \pm 90°$. However, the Enkin-Watson method will tend to give an artificial solution to such data sets with lower inclinations and very low precision. Such soultions depend critically on the assumptions of the calculation method, and we feel that in such cases it may be better for scientists to have a clear indication that the information to separate (I, κ) is permanently lost.

2 The Studied Methods

Kono (J. Geophys. Res., 85, 3878-3882, 1980) presented a method based on equating the expectation values of $\cos\theta$ and $\cos^2\theta$ of the distribution to the data. In principle this is a correct method of moments estimation, and asymptotically unbiased as $N \rightarrow \infty$. For some examples of steep and dispersed inclinations the method breaks down.

McFadden and Reid (Geophys. J. R. Astron. Soc., 69, 307-319, 1982) critisized the assumptions of Kono and suggested a different solution to the maximum likelihood problem. Unfortunately there was an error in one of their equations. Although this error was early on modified by several workers, some paleomagnetists are still using the erroneous original method. Even with the correction the McFadden-Reid method is based on approximations that turn out to be inappropriate for dispersed steep inclinations and lead to inclination biases.

Enkin and Watson (Geophys. J. Int., 126, 495-504, 1996) presented a new approach. They weighted the likelihood function with a Bayesian factor, so their method is not solution to the maximum likelihood problem. This weighing gives better constraints of the solution for very dispersed and steep inclinations. They presented three totally different methods depending on the dispersion and steepness of the data. These are 1) arithmetic mean, 2) gaussian estimate, 3) marginal likelihood. In this comparison we have focused on their gaussian estimation, but for the steepest data one should use the marginal likelihood. However, the marginal likelihood leads to increased inclination bias as they change to assymmetric confidence intervals.



The number of Science Citation Index (SCI) citations indicates that the most favourite inclination-only method among scientists may be the McFadden-Reid method.

Arason and Levi (2006, AGU Fall Meeting) have outlined a new method that directly evaluates the maximum likelihood estimates of mean inclination and precision parameter without previous assumptions. Evaluation of the likelihood function and its derivatives is very problematic. For ordinary paleomagnetic data the direct evaluation of these functions includes exponential elements that lead to an overflow in any ordinary programming language. We were successful in analytically cancelling these exponential elements from the functions, and by accurate evaluation of Bessel and other functions we are able to accurately calculate the location of the maximum of the likelihood function.

4 Conclusions

In this study we compared methods to estimate mean inclinations from inclination-only data.

Independent of methods, the paleomagnetic inclination shallowing bias of individual studies is usually well within the confidence interval of the mean. However, it is essential to estimate unbiased inclination means from inclination-only data, because the results are sometimes combined from several individual studies to increase resolution.

Unbiased estimates of other statistical parameters are usually not as important as the mean inclinations.

The comparisons of this study shows that the McFadden-Reid method, which appears to be widely used, often leads to significant inclination biases. The original method includes an error and appears to give mean inclinations almost identical to the arithmetic mean. A modification to the method is not much better. In our opinion the McFadden-Reid method should be abandoned.

Arason and Levi (2006) presented a new method, which accurately calculates the maximum likelihood estimates of mean inclinations from inclination-only data. This study indicates that this method results in the least inclination biases.

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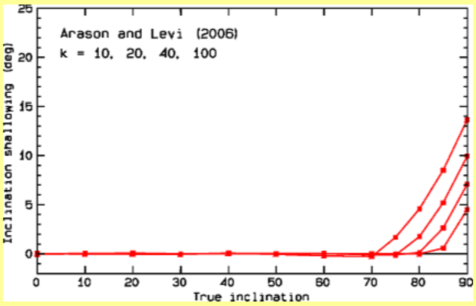
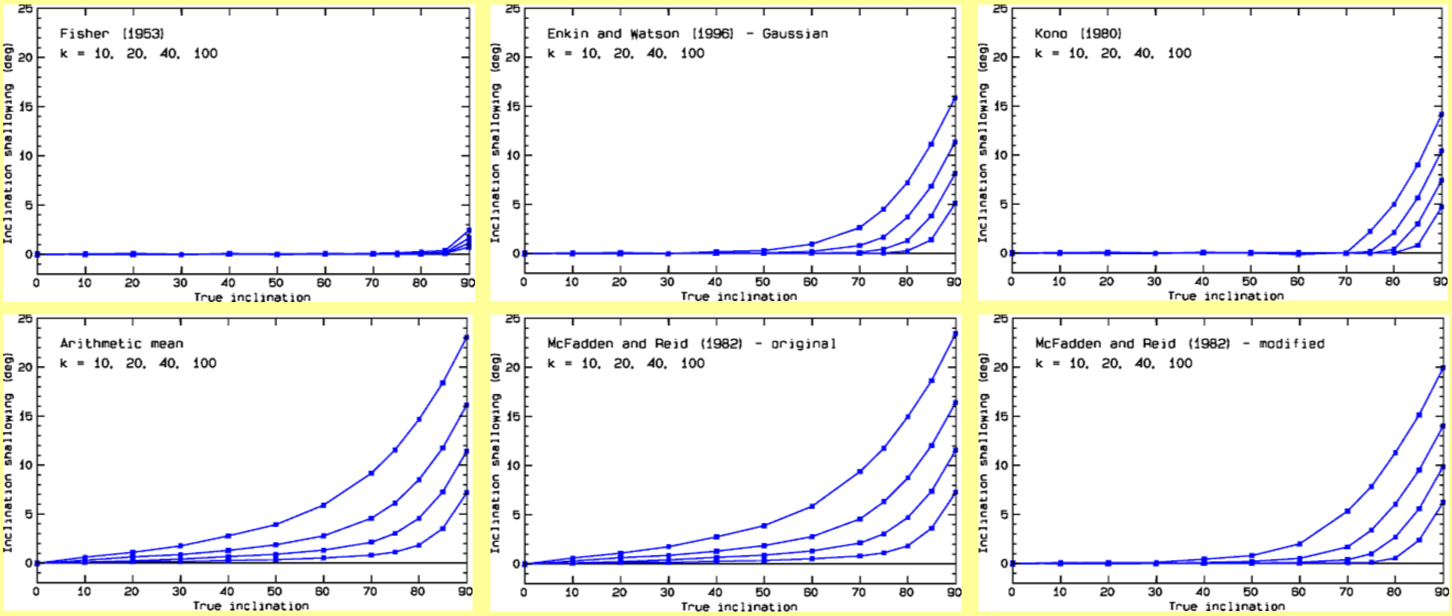
A The Inclination Shallowing Bias of the Methods

The average inclination shallowing bias of the seven methods is shown in these graphs versus true inclination. Each graph shows four curves representing precision parameter (κ) of 10, 20, 40, and 100 (least biased).

Each point was calculated as the average of 1000 estimates of the mean inclination by the methods from random Fisher-distributed data sets where the sample number was fixed at N = 100. The McFadden-Reid method(s) seem to be similar to the arithmetic mean. For the McFadden-Reid and Kono estimates a no-solution was not included. Although the Fisher-estimates are themselves unbiased, arithmetic average of the one thousand Fisher mean-inclinations for I = 90° is slightly biased due to the arithmetic mean bias.

For comparison purposes the solutions on the edge (I = 90°) by the Arason-Levi method were not included in the means. Inclusion of these vertical solutions would decrease the bias of the method.



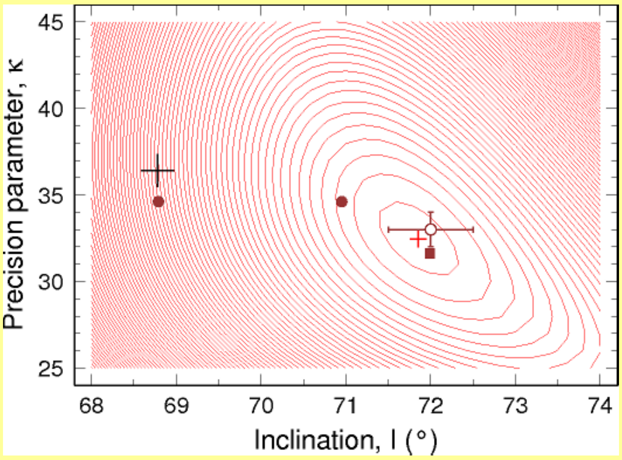


B Fishers Numerical Example

Table. *Different Methods Used to Estimate Directional Data Statistics**

Method	*Average Declination, deg*	*Average Inclination, deg*	*Precision Parameter*	*95% Confidence Limits, deg*
Arithmetic mean	22.87	68.78	36.42	7.48
Fisher [1953]	24.27	70.89	35.08	8.81
		Inclination-only:		
Briden and Ward [1966]	—	72	33	—
Kono [1980]	—	71.99	31.64	9.29
McFadden and Reid [1982]				
Original	—	68.79	34.62	9.25
Modified	—	70.95	34.62	−2.16 ± 9.24
Enkin and Watson [1996]				
Gaussian estimate	—	71.87	25.13	7.47
Marginal likelihood	—	70.9	22.93	+14.0 / −7.0
Arason and Levi [2006]	**—**	**71.85**	**32.45**	**9.17**
Direct Maximum Likelihood**				
Mathematica®	—	71.85	32.45	—
Maple®	—	71.85	32.45	—

* All the listed inclination-only methods are attempting to evaluate the maximum likelihood estimates, except the Enkin-Watson method, which weights the likelihood function by a Bayesian factor.
** Direct location of the maxima of the exact form of the likelihood function by comprehensive mathematical and statistical software packages.



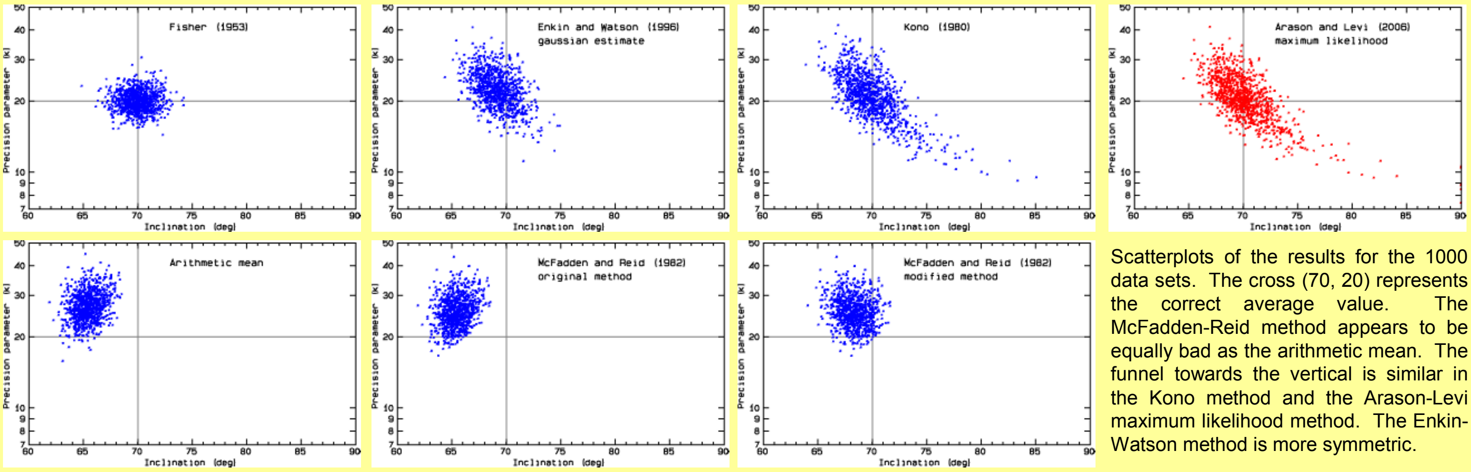
Contours of the log-likelihood function for the inclination data, and results of various methods to identify its peak: Black **+** *represents the arithmetic mean; Open circle with error bars the Briden-Ward graphical method; Filled circles McFadden-Reid, both their original method, close to the arithmetic mean, and the modified method; Square the results of the Kono method; And the red* **+** *represents the results of the Arason-Levi method, which we claim to represent an accurate estimate of the maximum likelihood.*

In this sample data we use the paleomagnetic data used in a numerical example by *Fisher* (1953). His nine inclinations were: 66.1, 68.7, 70.1, 82.1, 79.5, 73.0, 69.3, 58.8, and 51.4.

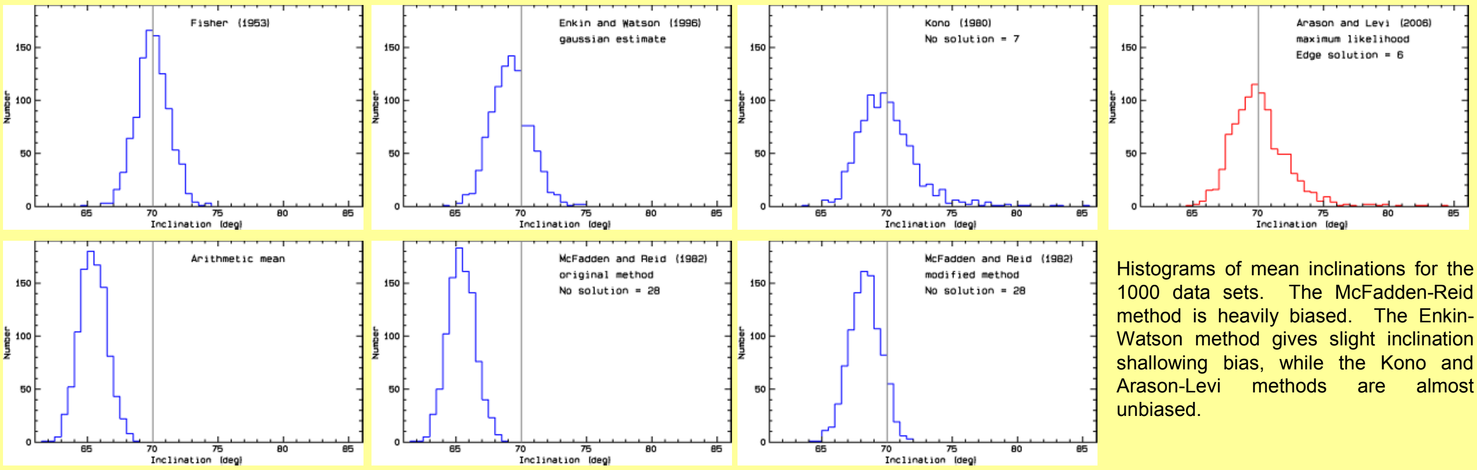
The maximum likelihood method of Arason and Levi gives identical values to the direct evaluation of the mathematical and statistical packages, Mathematica and Maple. At least for this data sample, the Arason-Levi method accurately evaluates the maximum likelihood estimates for inclination-only data.

C Example of one of the 160 combinations: *I* = 70°, *κ* = 20, *N* = 100

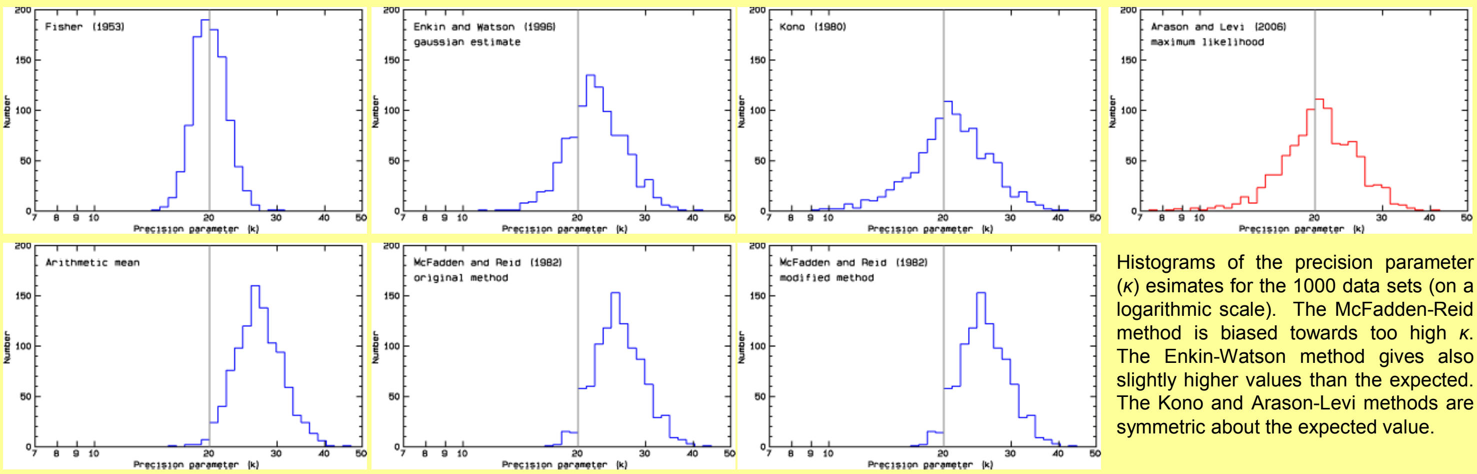
For the combination of true inclination *I* = 70°, precision parameter *κ* = 20 and sample number *N* = 100, we generated one thousand random Fisher-distributed data sets, and for these we calculated the true Fisher mean, also using declinations. For inclination-only data the mean was calculated using six methods including the maximum likelihood estimates by our new robust technique. The following graphs show the distribution of the 1000 estimates by the methods.



Scatterplots of the results for the 1000 data sets. The cross (70, 20) represents the correct average value. The McFadden-Reid method appears to be equally bad as the arithmetic mean. The funnel towards the vertical is similar in the Kono method and the Arason-Levi maximum likelihood method. The Enkin-Watson method is more symmetric.



Histograms of mean inclinations for the 1000 data sets. The McFadden-Reid method is heavily biased. The Enkin-Watson method gives slight inclination shallowing bias, while the Kono and Arason-Levi methods are almost unbiased.



Histograms of the precision parameter (*κ*) esimates for the 1000 data sets (on a logarithmic scale). The McFadden-Reid method is biased towards too high *κ*. The Enkin-Watson method gives also slightly higher values than the expected. The Kono and Arason-Levi methods are symmetric about the expected value.