The Maximum Likelihood Solution to Inclination-only Data

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1 Abstract

The arithmetic means of inclination-only data are known to introduce a shallowing bias. Several methods have been proposed to estimate unbiased means of the inclination along with measures of the precision.

Most of the inclination-only methods were designed to maximize the likelihood function of the marginal Fisher distribution. However, the exact analytical form of the maximum likelihood function is fairly complicated, and all these methods require various assumptions and approximations that are inappropriate for many data sets. For some steep and dispersed data sets, the estimates provided by these methods are significantly displaced from the peak of the likelihood function to systematically shallower inclinations. The problem in locating the maximum of the likelihood function is partly due to difficulties in accurately evaluating the function for all values of interest. This is because some elements of the log-likelihood function increase exponentially as precision parameters increase, leading to numerical instabilities.

In this study we succeeded in analytically cancelling exponential elements from the likelihood function, and we are now able to calculate its value for any location in the parameter space and for any inclination-only data set, with full accuracy. Furtermore, we can now calculate the partial derivatives of the likelihood function with desired accuracy. Locating the maximum likelihood without the assumptions required by previous methods is now straight forward.

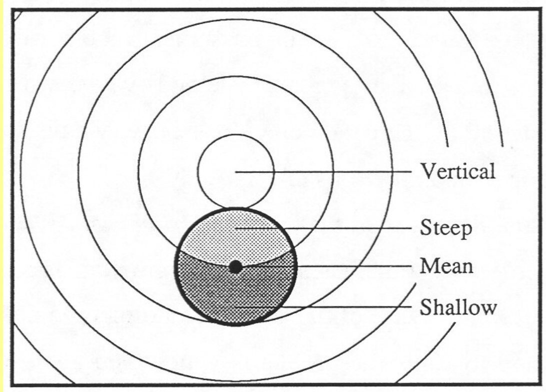
The information to separate the mean inclination from the precision parameter will be lost for very steep and dispersed data sets. It is worth noting that the likelihood function always has a maximum value. However, for some dispersed and steep data sets with few samples, the likelihood function takes its highest value on the boundary of the parameter space, i.e. at inclinations of ±90°, but with relatively well defined dispersion. Our simulations indicate that this occurs quite frequently for certain data sets, and relatively small perturbations in the data will drive the maxima to the boundary. We interpret this to indicate that, for such data sets, the information needed to separate the mean inclination and the precision parameter is permanently lost.

To assess the reliability and accuracy of our method we generated large number of random Fisher-distributed data sets and used seven methods to estimate the mean inclination and precision paramenter. These comparisons are described by ***Levi and Arason*** (2007, IUGG meeting). The results of the various methods is very favourable to our new robust maximum likelihood method, which, on average, is the most reliable, and the mean inclination estimates are the least biased toward shallow values.

Further information on our inclination-only analysis can be obtained from:

http://www.vedur.is/~arason/paleomag

2 Mean Bias of Inclination-only Data



The geometry of the sphere dictates that any circularly symmetric distribution about a true mean will be represented by more shallow inclinations than steep as compared to the mean. Arithmetic average of inclinations will therefore result in a too shallow estimate of the mean. From ***Arason*** *(Ph.D. Thesis, Oregon State University, 1991, Fig. 5.1, p. 207).*

The "normal" distribution of three dimensional directions is the Fisher-distribution. By using Fisher-statistics one can obtain unbiased directional mean of a sample drawn from such a distribution (***Fisher***, Proc. R. Soc. London, Ser. A, 217, 295-305, 1953). Sometimes one has only access to inclinations and not declinations. Paleomagnetic directions from borecores usually lack declinations, but inclinations can be reliable. ***Briden and Ward*** (Pure Appl. Geophys., 63, 133-152, 1966) showed that for such inclination-only data, the arithmetic mean is biased toward shallow inclinations.

In paleomagnetic applications this inclination shallowing bias is usually less than a few degrees. For individual studies such a discrepancy is of a minor importance and usually well within the confidence limits of the study. However, since this is a one sided bias, attempts to combine results of many studies may lead to errors. Therefore, improper procedures for estimating mean inclinations in individual studies can seriously affect combined average estimates.

3 The problem and our Solution

Over forty years ago ***Briden and Ward*** (1966) pointed out that the arithmetic mean of inclination-only data introduces a shallowing bias. Furthermore, they derived the likelihood function assuming the directions follow the Fisher-distribution, and presented a graphical method to estimate the true mean inclination along with the precision parameter (κ).

The likelihood function includes exponential elements, that are very difficult to accurately evaluate.

Several workers have attempted to derive a method to calculate the maximum likelihood estimates of mean inclination and the precision (e.g., ***Kono***, J. Geophys. Res., 85, 3878-3882, 1980; ***McFadden and Reid***, Geophys. J. R. Astron. Soc., 69, 307-319, 1982). Those methods make certain assumptions and approximations, which sometimes are inappropriate leading to inaccurate estimates of the maximum likelihood, and a bias toward shallow inclinations.

In this study we present a simple and robust method to calculate simultaneously the maximum likelihood estimates of the mean inclination and precision parameter without the assumptions and approximations of previous workers.

The exact mathematical form of the log-likelihood function and its derivatives is available. The task is to accurately identify the pair of (θ, κ) that maximize the likelihood function.

However, there are two major obstacles in directly identifying the maximum:

1. There are exponential elements in the likelihood function that become impossible to directly evaluate, and attempts in ordinary programming languages will often lead to an overflow or very inaccurate values, even for ordinary paleomagnetic data.

2. The likelihood function and its derivatives include Bessel functions that are difficult to accurately evaluate.

Our direct solution to the maximum likelihood problem includes:

A. We were successful in analytically cancelling all the exponential terms from the log-likelihood function.

B. We use an accurate estimation of the Bessel functions, many orders of magnitude more accurate than previous attempts on the problem.

Once these obstacles are cleared, accurate calculation of the maximum is straight forward.

4 Conclusions

The problem of estimating unbiased means of paleomagnetic inclination-only data was described over forty years ago.

Several methods have been proposed to solve the problem. Some of these methods have evaluated the maximum likelihood estimates. However, these methods are based on various approximations and assumptions that turn out to be inappropriate for steep and dispersed data. Unfortunately, these estimates are sometimes inaccurate and on average biased toward shallow inclinations.

Analytical cancellations of exponential elements in the functions of the problem are essential to calculate the estimates accurately.

We present a method with accurate representations of the functions needed to solve the problem.

The method that we present makes it possible for scientists to accurately calculate the maximum likelihood estimates of the inclination-only problem.

Detailed information on our inclination-only analysis can be obtained from:

http://www.vedur.is/~arason/paleomag

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